

Short note on energy maximization property of the first eigenfunction of the Laplacian

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Abstract

We consider the Dirichlet-energy maximization problem of the solution u_f of (2), among all functions $f \in L^2(D)$, such that $\|f\|_2 = 1$. We show that the two maximizers are the first eigenfunctions of the Laplacian with Dirichlet boundary condition $f = \pm u_1$.

One of the classical problems in rearrangements theory is the minimization/maximization of the functional

$$\Phi(f) = \int_D |\nabla u_f|^2 dx, \quad (1)$$

where u_f is the unique solution of the Dirichlet boundary value problem

$$\begin{cases} -\Delta u_f = f & \text{in } D, \\ u_f = 0 & \text{on } \partial D, \end{cases} \quad (2)$$

and f belongs to the rearrangement class.

Let us change the problem and maximize the functional (1) over the unit sphere in L^2 , i.e.,

$$f \in \mathcal{S} = \{f \in L^2(D) : \|f\|_2 = 1\}.$$

First we relax the problem and consider the maximization of $\Phi(f)$ over the unit ball

$$\mathcal{B} = \{f \in L^2(D) : \|f\|_2 \leq 1\}$$

in $L^2(D)$. The existence of the maximizer(s) $\hat{f} \in \mathcal{S}$ follows from weak closedness and convexity of \mathcal{B} , and strict convexity and weak continuity of Φ .

Passing to the limit in the extremality condition

$$t^{-1} [\Phi(\hat{f} + t(f - \hat{f})) - \Phi(\hat{f})] \leq 0$$

we obtain

$$\langle \Phi'(\hat{f}), f - \hat{f} \rangle \leq 0.$$

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Observe that $\Phi'(\hat{f})$ can be associated with $2\hat{u} := 2u_{\hat{f}}$,

$$\begin{aligned} \epsilon^{-1} \int_D |\nabla u_{f+\epsilon h}|^2 - |\nabla u|^2 dx = \\ \int_D \nabla(u_{f+\epsilon h} + u_f) \cdot \nabla u_h dx \xrightarrow{\epsilon \rightarrow 0} \int_D 2u_f h dx. \end{aligned} \quad (3)$$

Thus,

$$\int_D f \hat{u} \leq \int_D \hat{f} \hat{u} dx$$

for all $f \in \mathcal{B}$.

Applying Cauchy-Schwartz inequality we see that

$$\int_D \hat{f} \hat{u} dx \leq \|\hat{u}\|_2 \|\hat{f}\|_2 \leq \|\hat{u}\|_2$$

and the equality holds if and only if $\hat{f} = \lambda \hat{u}$. Thus, \hat{f} must coincide with one of the eigenfunctions $u_k \in \mathcal{S}$ of the eigenvalue problem with Dirichlet boundary conditions (see [1])

$$\begin{cases} -\Delta u = \lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \quad (4)$$

and $\hat{u} = \lambda_k^{-1}(D)u_k$, where λ_k is the k th eigenvalue.

The maximization of Φ over \mathcal{B} reduces now to the maximization of

$$\Phi(u_k) = \lambda_k^{-2}(D) \int_D |\nabla u_k|^2 = \frac{1}{\lambda_k(D)}$$

over k , which happens when $k = 1$.

We have proven the following theorem

Theorem 0.1. *The solutions of the maximization problem*

$$\max_{\|f\|_2 \leq 1} \int_D |\nabla u_f|^2 dx,$$

where the function u_f is the solution of the equation (2), are the first eigenfunctions $\hat{f} = \pm u_1 \in \mathcal{S}$ of the Dirichlet eigenvalue problem (4),

$$\Phi(\hat{f}) = \frac{1}{\lambda_1(D)},$$

and $\hat{u} = \pm \lambda_1^{-1}(D)u_1$.

Remark 0.1. *To see that the minimization of Φ over \mathcal{S} does not have a solution we need only to take a sequence $f_k \in \mathcal{S}$, such that $f_k \rightarrow 0$ and see that $\Phi(\hat{u}_k) \rightarrow 0$. Observe that $f = 0$ is the minimizer of Φ over \mathcal{B} .*

References

- [1] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.